Use Laplace transforms to solve the initial value problem

SCORE: _____ / 14 PTS

$$y'' + 4y' + 4y = 13e^t \cos 2t, \quad y(0) = 2, \quad y'(0) = -3$$

$$s^2Y - sy(0) - y'(0) + 4(sY - y(0)) + 4Y = \frac{13(s-1)}{(s-1)^2+4}$$

$$(s^2+4s+4)Y - 2s - 5 = \frac{13(s-1)}{(s-1)^2+4}$$

$$Y = \frac{2s+5}{(s+2)^2} + \frac{13(s-1)}{(s+2)^2[(s-1)^2+4]}$$

$$\frac{2s+5}{(s+2)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$$

$$2s+5 = A(s+2) + B$$

$$A = 2, \ B = 1$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s+2} + \frac{1}{(s+2)^2}\right\} = 2e^{-2t} + te^{-2t}$$

$$\mathcal{L}^{-1}\left\{\frac{-\frac{5}{13}}{s+2} + \frac{-3}{(s+2)^2} + \frac{\frac{5}{13}(s-1) + \frac{12}{13}(2)}{(s-1)^2+4}\right\}$$

$$= -\frac{5}{13}e^{-2t} - 3te^{-2t} + \frac{5}{13}e^t\cos 2t + \frac{12}{13}e^t \sin 2t$$

$$y = \frac{21}{13}e^{-2t} - 2te^{-2t} + \frac{5}{13}e^t\cos 2t + \frac{12}{13}e^t \sin 2t$$

SANITY CHECK: $s=-3$

$$\frac{13(-4)}{20} \overset{?}{=} \frac{5}{13} - 3 + \frac{-\frac{20}{13}+\frac{24}{13}}{20}$$

$$-\frac{13}{5} \overset{?}{=} \frac{5}{13} - 3 + \frac{1}{65} = \frac{26}{65} - 3$$

$$= \frac{2}{5} - 3 = \frac{-13}{5} \checkmark$$

$$\frac{13(s-1)}{(s+2)^2[(s-1)^2+4]} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C(s-1)+D(2)}{(s-1)^2+4}$$

$$13(s-1) = A(s+2)[(s-1)^2+4] + B[(s-1)^2+4] + C(s-1)(s+2)^2 + D(2(s+2)^2)$$

$$s=-2: \ -39 = 13B \rightarrow B = -3$$

$$13(s-1) = A(s+2)[(s-1)^2+4] - 3s^2+6s-15 + C(s-1)(s+2)^2 + D(2(s+2)^2)$$

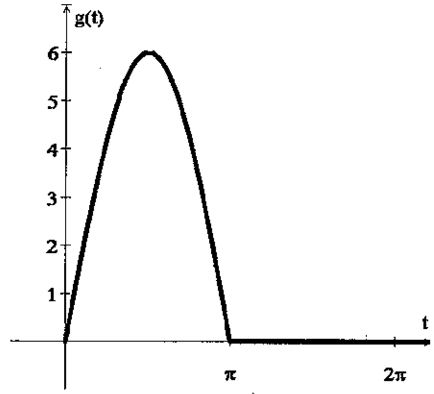
$$3s^2+7s+2 = (\cancel{s+2})(3s+1) = A(\cancel{s+2})[(s-1)^2+4] + C(s-1)(s+2)^{\cancel{2}} + D(2(s+2)^{\cancel{2}})$$

$$s=-2: \ -5 = 13A \rightarrow A = -\frac{5}{13}$$

$$s=1: \ 4 = 4A + 6D \rightarrow D = \frac{2}{3}(1-A) = \frac{12}{13}$$

COEF OF s^2: $0 = A + C \rightarrow C = -A = \frac{5}{13}$

Use Laplace transforms to solve the initial value problem $y'' + y' - 2y = g(t)$, $y(0) = -5$, $y'(0) = 4$ **SCORE: _____ / 16 PTS**
where $g(t)$ is the function whose graph is shown on the right.



<u>NOTE: The first piece of $g(t)$ is a half-period of a sinusoidal function. The second piece is 0.</u>

$$\begin{aligned} &s^2Y - sy(0) - y'(0) \\ &+sY \quad - y(0) \\ &-2Y \end{aligned} = \frac{6}{s^2+1} - 6e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\}$$

$\sin(t+\pi) = -\sin t$

$$g(t) = \begin{cases} 6\sin t, & t \in [0,\pi] \\ 0, & t > \pi \end{cases}$$
$= 6\sin t - 6u(t-\pi)\sin t$

$$(s^2+s-2)Y + 5s + 1 = \frac{6}{s^2+1} + \frac{6e^{-\pi s}}{s^2+1}$$

$$Y = -\frac{5s+1}{(s+2)(s-1)} + \frac{6}{(s+2)(s-1)(s^2+1)} + e^{-\pi s}\frac{6}{(s+2)(s-1)(s^2+1)}$$

$$-\frac{5s+1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

$-5s - 1 = A(s-1) + B(s+2)$

$s=1$: $-6 = 3B \rightarrow B = -2$

$s=-2$: $9 = -3A \rightarrow A = -3$

SANITY CHECK $s=2$: $-\frac{11}{4} \stackrel{?}{=} \frac{-3}{4} - 2$ ✓

$$\mathcal{L}^{-1}\left\{\frac{-3}{s+2} + \frac{-2}{s-1}\right\} = -3e^{-2t} - 2e^{t}$$

$$\frac{6}{(s+2)(s-1)(s^2+1)} = \frac{C}{s+2} + \frac{D}{s-1} + \frac{Es+F}{s^2+1}$$

$$6 = C(s-1)(s^2+1) + D(s+2)(s^2+1) + Es(s+2)(s-1) + F(s+2)(s-1)$$

$s=1$: $6 = 6D \rightarrow D = 1$

$s=-2$: $6 = -15C \rightarrow C = -\frac{2}{5}$

$s=0$: $6 = \frac{2}{5} + 2 - 2F \rightarrow F = -\frac{9}{5}$

COEF OF s^3: $0 = C + D + E \rightarrow E = -(C+D) = -\frac{3}{5}$

SANITY CHECK $s=2$: $\frac{6}{4(5)} \stackrel{?}{=} \frac{-\frac{2}{5}}{4} + 1 + \frac{-\frac{6}{5} + -\frac{9}{5}}{5}$

$\frac{3}{10} \stackrel{?}{=} -\frac{1}{10} + 1 - \frac{3}{5}$

$= \frac{-1+10-6}{10}$ ✓

$$\mathcal{L}^{-1}\left\{\frac{-\frac{2}{5}}{s+2} + \frac{1}{s-1} + \frac{-\frac{3}{5}s - \frac{9}{5}}{s^2+1}\right\} = -\frac{2}{5}e^{-2t} + e^{t} - \frac{3}{5}\cos t - \frac{9}{5}\sin t$$

$$y = -\frac{17}{5}e^{-2t} - e^{t} - \frac{3}{5}\cos t - \frac{9}{5}\sin t + u(t-\pi)\left[-\frac{2}{5}e^{-2(t-\pi)} + e^{t-\pi} + \frac{3}{5}\cos t + \frac{9}{5}\sin t\right]$$

SINCE $\cos(t-\pi) = -\cos t$, $\sin(t-\pi) = -\sin t$